TOWARDS A GENERAL SOLUTION OF THE LINEAR HEISENBERG PROBLEM

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Our presentation of the general solution of the linear Heisenberg problem is based on two approaches: The representation of this solution in terms of the Bethe-Hulthén scheme and its asymptotic form in terms of strings. This asymptotic form is supposed to be complete. The relation between these approaches is explained.

1. Introduction

The construction of a *complete set* of stationary states of the linear Heisenberg system with periodic boundary conditions (rings) has been a topic of intensive research for many decades. Many eminent theoreticians have made their contribution to this joint effort, but we should mention in the first place the epoch-making work of Bethe [1] and Hulthén. [5] Their work is the basis of many papers on this topic that have appeared during the past 70 years.

We do no have the intention to give a complete survey of all this work but we will try to indicate that especially an *asymptotic approach* starting from the *Hypothesis of Strings* gives the prospect of a *general* and *complete* solution of the Heisenberg chain. [7]

In section 2 we first give a transformation of the general Bethe-Hulthén solution (BHS) into a form that is suitable for the formulation of the Hypothesis of Strings [7] and show that this transformation in first instance results in a simplification in the sense that the two-fold set of parameters $(k_j, \phi_{j,l})$ in BHS reduces to a single one in terms of the parameters Λ_j. For a presentation of the solution of the Heisenberg problem, however, BHS has the advantage of a picture of simple waves, which show the phenomenon of *reflection* without *distortion.*

The asymptotic form of the solution, the string solution (SS), has a simple presentation in terms of the parameter set $\{\Lambda_j\}$, and this is the contents of section 3, for the special case of one string. The generalization for more strings is explained in section 4. Having found these asymptotic solutions one may determine the corresponding BHS for a fixed but large number of spins N and find the corresponding set of integers $\{\lambda_j\}$ connected with the periodic boundary conditions. These integers classify a *family* of solutions for variable number of spins but a fixed number of inversions or deviations, i.e. *upturned* spins in a ferromagnetic reference state.

In section 5 we present a general method to derive the asymptotic solutions and demonstrate the relation with a corresponding BHS. The general idea is that all possible combinations of strings lead to an asymptotic solution and that the set of string solutions is complete.

We give in section 6 three simple examples of asymptotic solutions within the BHS and their relation with SS.

In several papers by other authors special cases are discussed in detail. [2,3,4,6,8]

Following a BHS in lowering N one observes a profusion of interesting algebraic phenomena, which will be discussed in some detail in the Appendix of this paper. Intensive numerical research during the past decade gave us the possibility to present the following general picture:

- for special values of N some sets $\{\lambda_n\}$ become redundant in the sense that they represent solutions that are also represented by other sets. This occurs if one of the quantities $|\mathrm{Re}\, k_j|$ or $|\mathrm{Re}\, \phi_{j,k}|$ reaches the value π. It turns out that these quantities may always be chosen within the interval $[0, \pi]$ for well-chosen sets $\{\lambda_n\}$. This will be clear after a careful look at the equations for k_j and $\phi_{j,k}$ given in the next section. The corresponding N values may be called *transition* points,

- for special values of N an originally *real* pair of wave numbers may change into a *complex conjugated* pair. These N values are called *critical* points,

- for special values of N the imaginary part of a wave number may become singular. This always occurs for pairs of complex conjugated wave numbers. For these N the corresponding BHS finds its *limit* point and for lower numbers of spins this solution need not be considered.

2. The Bethe-Hulthén solution in terms of the parameters $\boldsymbol{\Lambda_j}$

The BHS for a ring of N spins with the standard Heisenberg Hamiltonian for nearest neighbours interaction (see for example Takahashi [7], formula

2.1 with substitutions $J = 4$, $H = 0$) has the general form:

$$\Psi = \sum_{1\leq j_1\leq\ldots\leq j_r\leq N} \sum_P \exp\left[i\left(\sum_{l=1}^{r} k_{P(l)} j_l + \frac{1}{2}\sum_{m<n} \phi_{P(m),P(n)}\right)\right]\Phi_{j_1 j_2 \ldots j_r}, \quad (1)$$

in which the symbol $\Phi_{j_1 j_2 \ldots j_r}$ denotes a state with r deviations from the ferromagnetic reference state, which deviations are at the positions $j_1, j_2, \ldots j_r$. The states with deviations at fixed positions are combined linearly to form a state Ψ for which the deviations have a wavelike character and are represented by the wave numbers or quasi momenta k_l. All permutations P of the wave numbers over the ordered deviations occur in (1), each permutation corresponding with a special phase. In the BHS the quasi momenta k_l and the phases $\phi_{l,m}$ obey a set of equations with r *fixed integers* $\{\lambda_j\}$,which identify a particular solution:

$$Nk_l = 2\pi\lambda_l + \sum_{m(\neq l)} \phi_{l,m}, \quad (l, m = 1, 2..r), \quad (2)$$

$$-\frac{N}{2} \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_r \leq \frac{N}{2}, \quad (3)$$

and:

$$\cot(\frac{\phi_{l,m}}{2}) = \frac{\sin(\frac{k_l - k_m}{2})}{\cos(\frac{k_l + k_m}{2}) - \cos(\frac{k_l - k_m}{2})}. \quad (4)$$

It should be understood that the quasi momenta and the phases may have complex values and in special cases may be *singular.*

The BHS given in (1-4) may be transformed into a form suitable for the formulation of the SS, if we make use of the following substitution:

$$\Lambda_l = \cot(\frac{k_l}{2}), \quad (5)$$

which is equivalent to:

$$\exp(ik_l) = \frac{\Lambda_l + i}{\Lambda_l - i} = e(\Lambda_l) = [e(-\Lambda_l)]^{-1}, \quad (6)$$

and from (4) and (5) one may derive:

$$\exp(i\phi_{l,m}) = e(\frac{\Lambda_l - \Lambda_m}{2}). \quad (7)$$

In (6) the definition of the function $e(x)$ is given.

After multiplication by:

$$\prod_{l<m} \exp(i\phi_{l,m}) \quad (8)$$

the formula (1) may be transformed into:

$$\Psi = \sum_{1\leq j_1<j_2<\ldots j_r\leq N} \sum_P \prod_{l=1}^{r} e(\Lambda_{P(l)})^{j_l} \prod_{\substack{1\leq m<n\leq r \\ P(m)>P(n)}} e(\frac{\Lambda_{P(m)}-\Lambda_{P(n)}}{2})\Phi_{j_1 j_2 \ldots j_r}, \quad (9)$$

with the condition:

$$e(\Lambda_l)^N = \prod_{\substack{m=1 \\ m(\neq l)}}^{r} e(\frac{\Lambda_l-\Lambda_m}{2}), \quad (10)$$

which follows from (2).

Because of the fact that the energy of all stationary states should be *real* it stands to reason that the quasi-momenta for any solution of the B.H. - problem should appear in *complex conjugated* pairs and the same may be said of the parameters Λ_i.

3. Solution with one single string

The *translational invariance* of an infinite chain suggests to explore the possible existence of a stationary state that corresponds with a *localized excitation* or a *wave packet* that moves along the chain without changing its form. In the asymptotic regime of large N this would result in a solution of (10) *independent* of the number of particles. This condition is the key to the two possibilities:

$$e(\Lambda_l)^N \Rightarrow 0 \text{ and } e(\Lambda_l)^N \Rightarrow \infty. \quad (11)$$

In both cases the imaginary part of Λ_l should be different from 0, according to (6). The consequence of the existence of a limit 0 for a given l, however, is that at least *one* of the factors in the right member of (10) is 0. But the condition:

$$e(\frac{\Lambda_l-\Lambda_m}{2}) = 0, \quad (12)$$

implies:

$$\frac{\Lambda_l-\Lambda_m}{2} = -i, \quad (13)$$

as a consequence of (6).

Now we take the following *ordered* set of Λ_l, i.e. the one given by Takahashi:[7]

$$\Lambda_l = \Lambda + i(r+1-2l), \; l = 1,2,..r, \quad (14)$$

in which r is the *length* of the string, the row of spin inversions in the wave packet. This implies that (13) is true for $m = l-1$ and careful inspection of the right member of (9) shows that there is always such a factor in every term with the exception of the one with the natural order for which the permutation P is trivial, i.e.: $P(j) = j$. This leaves us with a simplified version of (9):

$$\Psi = \sum_{1 \leq j_1 < j_2 < \ldots j_r \leq N} \prod_{l=1}^{r} e(\Lambda_l)^{j_l} \Phi_{j_1 j_2 \ldots j_r}, \tag{15}$$

with Λ_j given by (14). It should be kept in mind that this is an *asymptotic* solution, as indicated in Takahashi's paper.

So far the impression could exist that the parameter Λ could be given any value, but the *reality* of the total energy and the total quasi momentum makes it plausible that the individual quasi momenta k_j appear in *complex conjugated pairs.* This is not a strict proof but the experience teaches us that so far only B.H. states have been found that obey this condition. Then it follows that according to (5) that also the Λ_j form complex conjugated pairs and, because of the special way we have chosen to represent the Λ_j in (14), the parameter Λ should be *real* and:

$$\Lambda_j = \Lambda^*_{r-j+1}. \tag{16}$$

The actual value of Λ may be derived from (6):

$$\exp(ik) = \exp i(k_1 + k_2 + \ldots k_r) = \prod_{j=1}^{r} \frac{\Lambda_j + i}{\Lambda_j - i} = \frac{\Lambda + ir}{\Lambda - ir}, \tag{17}$$

which turns out to be a complex number with modulus 1, as it should be, because the total quasi momentum k is real. From (17) we immediately derive a relation between Λ and k:

$$\cot(\frac{k}{2}) = \frac{\exp(ik) + 1}{\exp(ik) - 1} i = \frac{\Lambda}{r}. \tag{18}$$

The possible values of the total quasi momentum are determined by the well-known *periodic boundary condition:*

$$Nk = 2\pi\lambda, \;\; \lambda = \sum_{j=1}^{r} \lambda_j, \tag{19}$$

or:

$$\left(\frac{\Lambda + ir}{\Lambda - ir} \right)^N = 1, \tag{20}$$

as follows from (2) and (17). The value of Λ as a function of k immediately follows from (18).

Our method is hybrid in the sense that the final result follows from two different arguments, one is asymptotic in character whereas the other is just the well-known periodic boundary condition for a finite system.

It may be concluded that, for the special set of Λ_j given in (14) with *real* Λ, all $\phi_{l,m}$ may be chosen to be *purely imaginary*, as follows from (7). The same may be said for the differences $k_l - k_m$. Then it follows from (2) that in the B.H.-representation all parameters λ_l *within one string* should be equal.

4. Generalization: States with 2 or more strings.

A straightforward generalization of an asymptotic solution with *one* string can be made by just superposing *two or more strings* in the following way. We imagine a state with an asymptotic number of spins with two strings at a large distance:

$$\Psi =$$

$$\sum_{1\leq j_1<j_2<\ldots j_r\leq M}\ \sum_{P\leq l_1<l_2<\ldots l_s\leq R}\ \prod_{n=1}^{r}\prod_{m=1}^{s} e(\Lambda_n)^{j_n} e(\Omega_m)^{l_s}\Phi_{j_1j_2\ldots j_rl_1l_2\ldots l_s},$$

$$M \gg 1,\ P-M \gg 1,\ R-P \gg 1,\ N-R \gg 1, \tag{21}$$

$$\Lambda_n = \Lambda + i(r+1-2n),\ \ n = 1,2,..r \tag{22}$$

$$\Omega_m = \Omega + i(s+1-2m), m = 1,2,...s, \tag{23}$$

in which Λ and Ω are *two different real numbers.* This last condition is the same as the condition that the two strings can be interchanged *without* resulting phase factors that are either 0 or ∞, as follows from (7). The resulting phase factor for the complete interchange of the two strings is now easily determined and an asymptotic form of a state with two strings may be represented by a superposition of two states of the form (21), with co-efficients:

$$1 \text{ and } \prod_{n=1}^{r}\prod_{m=1}^{s} e(\frac{\Omega_m-\Lambda_n}{2}), \tag{24}$$

and instead of (10) we now have the condition:

$$\prod_{m=1}^{s} e(\Omega_m)^N = \prod_{n=1}^{r}\prod_{m=1}^{s} e(\frac{\Omega_m-\Lambda_n}{2}). \tag{25}$$

In this case one should observe the analogy with the superposition of two single deviations with *real wave vectors* in the B.H.-scheme, by making the substitution:

$$r = s = 1, \quad \Lambda_n = k_1, \quad \Omega_m = k_2, \quad e(\frac{\Omega_m - \Lambda_n}{2}) = \exp(\phi_{2,1}). \tag{26}$$

For the asymptotic states with 2 strings we also have a boundary condition of the type (20), which now takes the form:

$$\left(\frac{\Lambda + ir}{\Lambda - ir}\frac{\Omega + is}{\Omega - is}\right)^N = 1. \tag{27}$$

Introducing the partial sums for the k_j of both strings:

$$k_\Lambda = \sum_{n=1}^{s} k_n, \quad k_\Omega = \sum_{m=1}^{s} k_m, \tag{28}$$

the condition (27) may be written:

$$\exp[iN(k_\Lambda + k_\Omega)] = 1 \quad \text{or} \quad N(k_\Lambda + k_\Omega) = 2\pi\mu, \tag{29}$$

in which expression μ again is an integer.

A generalization of the foregoing argument for states with more strings may be readily made by considering products of strings with all possible permutations of the group of indices corresponding to the different strings. Here again one observes an analogy with states with single deviations with *real* k.

For an asymptotic number of spins and a given number of deviations one generally has a complicated picture of the stationary states, but there only exist two possible configurations for neighbouring deviations:

-either neighbouring deviations stick together in a bound states or

-or they form a scattering state.

In the first case one has a string that moves along the chain without changing its form, but only showing a phase shift per lattice distance. Such a string may contain more than just the two deviations considered. It has a *real* total k and it may be considered as a generalization of one single deviation. In the second case one has individual deviations corresponding to different strings. So the asymptotic string solution (SS) will show a great analogy with BHS in that sense that its general form of the former is isomorphous with the latter with the restriction to *real* k for all strings.

5. A family of solutions starting from an asymptotic configuration

Starting from the Bethe-Hulthén equations (2-4) one may consider the following asymptotic regime:

$$\lambda = \lambda_0 N \qquad \lambda, N \Rightarrow \infty \qquad \lambda_0 \text{ fixed}, \tag{30}$$

which represents a situation for which the k_j and $\phi_{j,k}$ approach asymptotic values corresponding with SS. In this regime the different strings appear as wave packets of a form independent of N, and with an overall complex phase that depends on the position of the packets in the chain. This phase follows from the string solution given in the foregoing section.

Given the SS one has all the relevant information of an asymptotic solution within the BH-scheme, i.e. the wave numbers k_j, from which the $\phi_{j,k}$ may be derived. Choosing now a sufficiently large N one may determine the "exact" solution starting from these asymptotic values of the wave numbers and the phases.This last solution is characterized by the well-known set $\{\lambda_j\}$,which have a simple relation with the an analogous set of integers of the string solution, which will be given below. The wave numbers and the phases may be considered now as a "quasi-continuous" function of N in lowering this number of spins. This procedure will be illustrated in the next section for three simple examples.

The SS, being similar to BHS with real λ_j, may be characterized by the integers μ_l, as follows from the analysis in the foregoing section. So the relation (25) may be rewritten:

$$NK_2 = 2\pi\mu_2 + \Phi_{2,1} \tag{31}$$

in which the symbol K_2 denotes the total wave number of the string (23) and $\Phi_{2,1}$ the total phase shift accompanying the exchange of the two strings. As stated before there exists a simple relation between the string parameters μ_l and the parameters λ_j of the constituents waves in the asymptotic regime:

$$\mu_l = \sum_j \lambda_j, \tag{32}$$

the sum refers to the λ_j of the corresponding string.

6. Three simple examples of string solutions

6.1. ($r = 2$, $\lambda_1 = \lambda_2 = \lambda/2$)

Within the SS the relevant equation is given in (18), which gives the value of the quantity Λ in terms of the total wave number k. The example we consider is the series:

$$(N = 1000, \lambda_1 = \lambda_2 = -40), (N = 200, \lambda_1 = \lambda_2 = -8),$$
$$(N = 100, \lambda_1 = \lambda_2 = -4). \tag{33}$$

The B.H.-equations (2-4) may be solved with the Ansatz:

$$k_1 = A - ib, \quad k_2 = A + ib, \quad \phi_{1,2} = -iq, \quad A = \pi\lambda/N, \tag{34}$$

for which the equations may be transformed into:

$$\sinh(\frac{q}{2})\sinh(\frac{q}{N}) + \cosh(\frac{q}{2})[\cos(\frac{\pi\lambda}{N}) - \cosh(\frac{q}{N})] = 0, \quad Nb = q. \tag{35}$$

The equations (34) and (35) give us the BHS for given N and λ. The only parameter to be determined in the string solution is Λ in this case, which follows from (18). The value of b for the string solution may be found with (6), which results in:

$$\exp(ik_1) = \frac{\Lambda_1 + i}{\Lambda_1 - i} \quad \exp(ik_2) = \frac{\Lambda_2 + i}{\Lambda_2 - i}, \tag{36}$$

$$\exp(2b) = \exp(ik_1)\exp(-ik_2) = \frac{\Lambda_1 + i}{\Lambda_1 - i} \times \frac{\Lambda_2 - i}{\Lambda_2 + i} =$$
$$= \frac{\Lambda + 2i}{\Lambda}\frac{\Lambda - 2i}{\Lambda} = \frac{\Lambda^2 + 4}{\Lambda^2}, \tag{37}$$

$$E = -\frac{16}{\Lambda^2 + 4}. \tag{38}$$

In Table I and II we give the results for the exact as well as the string solution and show the convergence of the BHS to the string solution for $N \Rightarrow \infty$.

Table I. Solutions for $\lambda_1 = \lambda_2 = -N/25$. Bethe-Hulthén scheme.

λ	λ_1	λ_2	N	A	b	q	E
-8	-4	-4	100	$-2\pi/25$	0.03417	3.41660	-0.24681
-16	-8	-8	200	$-2\pi/25$	0.03203	6.40600	-0.24736
-80	-40	-40	1000	$-2\pi/25$	0.03192	31.9209	-0.24739

Table II. String solution for $A = -2\pi/25$.

Λ	A	b	E
-7.78949	$-2\pi/25$	0.03192	-0.24739

N.B.: In the string solution the value of $q = \infty$. The single parameter for the string solution is $\mu = \lambda$ for this case.

6.2. $(r = 2, s = 1, \lambda_1 = \lambda_2 = \lambda/2, \lambda_3 = \lambda/4)$

Within the SS the relevant equations can be found in section 4 and we consider the cases:

$$\begin{gathered}(N = 1000, \lambda_1 = \lambda_2 = -40, \lambda_3 = -10),\\ (N = 200, \lambda_1 = \lambda_2 = -8, \lambda_3 = -2),\\ (N = 100, \lambda_1 = \lambda_2 = -4, \lambda_3 = -1).\end{gathered} \tag{39}$$

The B.H.-equations (2-4) may be solved with the Ansatz:

$$k_1 = A - ib, k_2 = A + ib, k_3 = n, \phi_{1,2} = -iq, \phi_{1,3} = P - ip, \phi_{2,3} = P + ip, \tag{40}$$

for which the equations may be transformed into:

$$\begin{gathered}NA = 2\pi\lambda_1 + P,\\ Nb = q + p,\\ Nn = 2\pi\lambda_3 - 2P\end{gathered} \tag{41}$$

and:

$$\begin{gathered}\sinh(q/2)\sinh(b) + \cosh(q/2)[\cos(A) - \cosh(b)] = 0,\\ \sin(P)[\cos(A-n) - \cosh(b)] +\\ +[\cos(P) + \cosh(p)][\sin(A) - \sin(n)\cosh(b) - \sin(A-n)] = 0,\\ \sinh(q)[\cos(A-n) - \cosh(b)] + [\cos(P) + \cosh(p)]\times\\ \times[-\cos(n)\sinh(b) + \sinh(b)] = 0.\end{gathered} \tag{42}$$

The solution of this set of equations may be found in Table III for the the three values of N given in (39).

For the two strings in the SS we have, according to (18):

$$\cot(k/2) = \cot(A) = \Lambda/2, \quad \cot(n/2) = \Omega, \tag{43}$$

or:

$$A = k/2 = \arctan(2/\Lambda), \quad b = 1/2\ln(1 + 4/\Lambda^2), \quad n = 2\arctan(1/\Omega), \tag{44}$$

in which the expression for b is essentially the same as the one given in (37). For the string solution we now have to give the single boundary condition (25):

$$\begin{aligned}e(\Omega)^N &= \prod_{n=1}^{2} e\left(\frac{\Omega - \Lambda_n}{2}\right) = \frac{\Omega - \Lambda + i}{\Omega - \Lambda - i} \times \frac{\Omega - \Lambda + 3i}{\Omega - \Lambda - 3i}\\ &= \exp(iv)\exp(iw) = \exp(i\Phi_{2,1}),\end{aligned} \tag{45}$$

from which it follows:

$$v = 2\arctan\left(\frac{1}{\Omega - \Lambda}\right), \quad w = 2\arctan\left(\frac{3}{\Omega - \Lambda}\right). \tag{46}$$

If we now treat the two strings as single waves in the BHS we have the boundary conditions:

$$\begin{aligned} &Nk = -v - w + 2\pi\mu_1 = \Phi_{1,2} + 2\pi\mu_1, \ \ \mu_1 \text{ integer}, \\ &Nn = v + w + 2\pi\mu_2 = \Phi_{2,1} + 2\pi\mu_2, \ \ \mu_2 \text{ integer}, \ \ \Phi_{1,2} = -\Phi_{2,1}. \end{aligned} \tag{47}$$

The equations (43), (46) and (47) now are a complete set of equations for the quantities k, n, Λ, Ω and w if we make a choice for the integers $\mu_1 = \lambda_1 + \lambda_2$ and $\mu_2 = \lambda_3$. The quantity b follows from (44). These string solution is given in Table IV

Table III. Solutions for $\lambda_1 = \lambda_2 = N/25, \lambda_3 = N/100$. Bethe-Hulthén scheme.

λ	λ_1	λ_2	λ_3	N	A	b	n	q
-9	-4	-4	-1	100	-0.24954	0.03379	-0.06641	3.36999
-18	-8	-8	-2	200	-0.25047	0.03181	-0.06455	6.35547
-90	-40	-40	-10	1000	-0.25116	0.03188	-0.06317	31.8711

λ	λ_1	λ_2	λ_3	N	P	p	E
-9	-4	-4	-1	100	0.17866	0.008545	-0.25218
-18	-8	-8	-2	200	0.17200	0.007417	-0.25403
-90	-40	-40	-10	1000	0.16698	0.006967	-0.25504

Table IV. String solution for $\lambda_1 = \lambda_2 = 4\lambda_3$.

λ	N	μ_1	μ_2	$A = k/2$	b
-9	100	-8	-1	-0.24954	0.03146
-18	200	-16	-2	-0.25047	0.03170
-90	1000	-80	-10	-0.25116	0.03188

λ	N	μ_1	μ_2	n	v	w
-9	100	-8	-1	-0.06641	-0.08979	-0.26793
-18	200	-16	-2	-0.06455	-0.08632	-0.25769
-90	1000	-80	-10	-0.06317	-0.08378	-0.25018

λ	N	μ_1	μ_2	Λ	Ω	E
-9	100	-8	-1	-7.84767	-30.10713	-0.25277
-18	200	-16	-2	-7.81739	-30.97210	-0.25406
-90	1000	-80	-10	-7.79489	-31.65217	-0.25504

6.3. $(r = 4, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda/4)$

In this example we make the supposition that the wave numbers constitute two complex conjugated pairs. Then the same can be said of the pairs $(\phi_{1,3}, \phi_{2,4})$ and $(\phi_{1,4}, \phi_{2,3})$:

$$\begin{aligned} &k_1 = A - ib, \ \ k_2 = A + ib, \ \ k_3 = C - id, \ \ k_4 = C + id, \\ &\phi_{1,2} = -ip, \ \ \phi_{1,3} = T - it, \ \ \phi_{1,4} = V - iv, \\ &\phi_{2,3} = V + iv, \ \ \phi_{2,4} = T + it, \ \ \phi_{3,4} = -ir. \end{aligned} \tag{48}$$

The Bethe-Hulthén equations now take the form:

$$\begin{aligned} &NA = 2\pi\lambda_1 + T + V, \ \ Nb = p + t + v, \ \ \lambda_1 = \lambda/4, \\ &NC = 2\pi\lambda_3 - T - V, \ \ Nd = -t + v + r, \ \ \lambda_3 = \lambda/4, \end{aligned} \tag{49}$$

$$\begin{aligned}&\sinh(p/2)\sinh(b)+\cosh(p/2)[\cos(A)-\cosh(b)]=0,\\&\sinh(r/2)\sinh(d)+\cosh(r/2)[\cos(C)-\cosh(d)]=0,\end{aligned}\tag{50}$$

$$\begin{aligned}&\frac{\sin(T)}{\cos(T)+\cosh(t)}+\frac{\sin(A)\cosh(d)-\sin(C)\cosh(b)-\sin(A-C)}{\cos(A-C)-\cosh(b-d)}=0,\\&\frac{\sinh(t)}{\cos(T)+\cosh(t)}+\frac{\cos(A)\sinh(d)-\cos(C)\sinh(b)+\sinh(b-d)}{\cos(A-C)-\cosh(b-d)}=0,\end{aligned}\tag{51}$$

$$\begin{aligned}&\frac{\sin(V)}{\cos(V)+\cosh(v)}+\frac{\sin(A)\cosh(d)-\sin(C)\cosh(b)-\sin(A-C)}{\cos(A-C)-\cosh(b+d)}=0,\\&\frac{\sinh(v)}{\cos(V)+\cosh(v)}+\frac{-\cos(A)\sinh(d)-\cos(C)\sinh(b)+\sinh(b+d)}{\cos(A-C)-\cosh(b+d)}=0.\end{aligned}\tag{52}$$

These equations have the solutions, presented in Table V, for a set of N values obeying $N=100\lambda/4$.

Table V. Bethe-Hulthén solutions for $\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda/4$, $N=100\lambda/4$.

$\lambda/4$	N	A	b	C	d	p
20	2000	0.06258	0.005926	0.06308	0.001991	0.6931
10	1000	0.06258	0.006054	0.06308	0.002012	0.6776
4	400	0.06249	0.007937	0.06318	0.002559	0.5105
1	100	0.06171	0.015033	0.06395	0.004881	0.2699

$\lambda/4$	N	T	t	V	v	r	E
20	2000	-0.4979	10.0602	-0.00003	1.0986	12.9435	-0.03142
10	1000	-0.2431	4.3025	-0.004247	1.0743	5.2402	-0.03141
4	400	-0.1215	1.8680	-0.01745	0.7964	2.0952	-0.03130
1	100	-0.09036	0.8218	-0.02152	0.4116	0.8983	-0.03059

To illustrate the concepts of *transition point* and *limit point* we also give here the results for a fixed set $\lambda_1=\lambda_2=\lambda_3=\lambda_4=1$ and variable N, i.e. from $N=100$ downwards. For a certain value of the number of spins the quantity $\mathrm{Re}(\phi_{1,3})=T$ reaches the value $-\pi$ and at that point the set of integers $\{\lambda_j\}$ can be replaced by an other one in order to keep $|T|$ within the boundaries $[0,\pi]$. The number of spins for which this "transition" occurs is not an integer, but it indicates a separation between two domains of integers N for which a different $\{\lambda_j\}$ set should be chosen. Finally one reaches a limit point for which the imiginary part of one wave number pair diverges: This is the natural ending of the solution with the given parameter set.

Table VI. Bethe-Hulthén solutions for the equivalent sets ($\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$) and ($\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \lambda_4 = 2$); $N^* = 10.3180$ is the transition point; $N = 8$ is the limit point.

λ_1	λ_2	λ_3	λ_4	N	A	b	C	d
1	1	1	1	100	0.06171	0.01503	0.06395	0.004881
1	1	1	1	15	0.3269	0.2978	0.5109	0.1435
1	1	1	1	$N*$	0.3045	0.5382	0.9134	0.4934
0	0	2	2	$N*$	0.3045	0.53819	0.9134	0.4934
0	0	2	2	9	0.1948	0.6344	1.2014	1.01886
0	0	2	2	8	0	0.66622	$\pi/2$	∞

λ_1	λ_2	λ_3	λ_4	N	T	t	V	v
1	1	1	1	100	-0.09036	0.8218	-0.02152	0.4116
1	1	1	1	15	-1.3034	2.7496	-0.07690	1.0480
1	1	1	1	$N*$	$-\pi$	3.7041	0	1.1310
0	0	2	2	$N*$	π	3.7041	0	1.1310
0	0	2	2	9	1.7270	3.9105	0.02643	1.1026
0	0	2	2	8	0	3.6010	0	1.0625

λ_1	λ_2	λ_3	λ_4	N	p	r	E
1	1	1	1	100	0.2699	0.8983	-0.03059
1	1	1	1	15	0.6700	3.8540	-1.0346
1	1	1	1	N^*	0.7178	7.6639	-1.7403
0	0	2	2	N^*	0.7178	7.6639	-1.7403
0	0	2	2	9	0.6967	11.9776	-1.9967
0	0	2	2	8	0.6662	∞	-2.1580

The string solution, which will be compared with the results of Table V, is fully determined by:

$$\begin{aligned} &r = 4, \;\; k = \frac{2\pi * 4}{100} = \frac{2\pi}{25}, \\ &\Lambda = 4\cot(\frac{\pi}{25}), \\ &\Lambda_1 = \Lambda + 3i, \;\; \Lambda_2 = \Lambda + 1i, \;\; \Lambda_3 = \Lambda - 1i, \;\; \Lambda_4 = \Lambda - 3i. \end{aligned} \tag{53}$$

The values for $\mathrm{Re}(k_j)$ ($j = 1,..4$) are all equal to $k/4 = \pi/50$. The imaginary parts follow from (6) and the the value of Λ given in (53):

$$\begin{aligned} &\Lambda = 31.6633, \\ &k_1 = A - ib, \;\; k_2 = A + ib, \;\; k_3 = C - id, \;\; k_4 = C + id, \\ &A = C = \pi/50, \;\; b = 0.005926, \;\; d = 0.001991. \end{aligned} \tag{54}$$

This result shows a reasonable accordance with Table V for the values for $N = 2000$. Only the quantity C shows a considerable discrepancy.

7. Concluding remark

As is well-known from the literature the representation of the solutions of the linear Heisenberg problem in terms of the BHS is *not unique.* In contradistinction the representation with strings always is, i.e. a representation with a set Λ, Ω, etc. one real number for each string. In the BHS one always may choose a $\{\lambda_j\}$ set for which the sets $\{|\mathrm{Re}\, k_j|\}, \{|\mathrm{Re}\, \phi_{j,l}|\}$ are in the interval $[0, \pi]$. This implies the change of the $\{\lambda_j\}$ set for certain values of N, which are in general non-integer. An example of such a change may be found in Table VI: For $N = 10.3180$ the set has to be changed to keep the $\{|\mathrm{Re}\, \phi_{j,l}|\}$ within the prescribed bounds. This special value of N we call a *transition point.* Another special value of N corresponds to a natural ending of a solution within the BHS, for which some quantities reach a *diverging imaginary part.* This occurs in the same example: For $N = 8$ the imaginary part of one pair of wave number diverges. Such a value of N we call a *limit point.* A third special point may occur for doublets with $\lambda_j = \lambda_{j+1} - 1$ for which an originally *real* pair (k_j, k_{j+1}) is transformated into a *complex conjugated* one. For this point the$\{\lambda_j\}$ set need not to be changed but the character of the BHS changes in a relevant way. The value of N represents a *critical point.*

These peculiarities of the BHS illustrate the drawbacks of this kind of representation, but it should be kept in mind that the changes are in effect not very fundamental because it only leads to a more convenient way of formulating a result, without changing its relevant properties, i.e. the values of the quantities Λ, Ω, etc.

Appendix: Special points in the BHS

a) Transition point

An example of such a point is found in our last example: $(r = 4, \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda/4)$. In this example the 4 wave numbers are two by two complex conjugated. The Bethe-Hulthén equations (48-52) were solved for the range 8 to 100 and it was observed that for $N = 10.3180$ the set $\{\lambda_j\}$ has to be changed in order to keep $|T|$, the modulus of real part of $\phi_{1,3}$ and $\phi_{2,4}$, within the proper bounds $[0, \pi]$. That such a transformation of the set $\{\lambda_j\}$ has the desired effect follows immediately from (49). These are the only equations of the total set in which the λ_j appear, and it should be clear that these numbers are not unique for a given solution, but could be manipulated in order to get a special form of this solution. This is a general feature of the BHS and it constitutes a drawback as compared to the solution with strings. For more complicated examples one may follow an analogous procedure to keep all the $|\mathrm{Re}\, k_j|$, $|\mathrm{Re}\, \phi_{j,k}|$ within the interval $[0, \pi]$.

b) Critical point

This special value of N appears for the case that two λ_j have a difference 1. Now we have a pair of wave numbers that are real (and different) for N above this critical value N_{cr} and form a complex conjugated pair below this number of spins. We illustrate this with two simple and related examples,which are represented by the following sets:

$$(\lambda_1 = -3, \lambda_2 = -2) \quad (\lambda_1 = -3, \lambda_2 = -2, \lambda_3 = 1). \tag{55}$$

For both examples we make, for sufficiently large N, the following transformation:

$$k_1 = A - b, \quad k_2 = A + b, \quad \phi_{1,2} = \pi - p, \tag{56}$$

whereas for the second example we have to add:

$$k_3 = n, \quad \phi_{1,3} = R - r, \quad \phi_{2,3} = R + r. \tag{57}$$

All quantities (A, b, p, n, R, r) are *real.*

The transformed B.H.-equations now take the form:
case I: $r = 2$, $(\lambda_1 = -3, \lambda_2 = -2)$.

$$NA = -5\pi, \tag{58}$$

$$Nb = p, \tag{59}$$

$$\cos(\frac{p}{2}) \sin(b) - \sin(\frac{p}{2})[\cos(b) - \cos(A)] = 0. \tag{60}$$

case II: $r = 3$, $(\lambda_1 = -3, \lambda_2 = -2, \lambda_3 = 1)$.

$$NA = -5\pi + R, \tag{61}$$

$$Nb = p + r, \tag{62}$$

$$Nk = 2\pi - 2R, \tag{63}$$

$$\cos(\frac{p}{2}) \sin(b) - \sin(\frac{p}{2})[\cos(b) - \cos(A)] = 0, \tag{64}$$

$$\begin{aligned} &\sin(R)[\cos(A - n) - \cos(b)] + \\ &+[\cos(R) + \cos(r)][\sin(A) - \sin(n)\cos(b) - \sin(A - n)] = 0, \end{aligned} \tag{65}$$

$$\begin{aligned} &\sin(r)[\cos(A - n) - \cos(b)] + \\ &+[\cos(R) + \cos(r)][-\cos(n)\sin(b) + \sin(b)] = 0. \end{aligned} \tag{66}$$

Typical for a critical point is the *degeneracy* of the spectrum of wave numbers, i.e. in case I: $k_1 = k_2$ or $b = p = 0$ and in case II: $k_1 = k_2$ or $b = p = r = 0$. This occurs for a special value of N, which we call N_{cr}. This is not an integer and it does not correspond to a *real, physical system* but it separates 2 regions of N values with quantitatively different properties of the BHS. In our explanation here we approach N_{cr} from the upper side.

To find N_{cr} we linearize in case I the equation (60) in terms of b and p and combine it with (59), which results in the set:

$$Nb - p = 0, \tag{67}$$

$$b - \frac{p}{2}[1 - \cos(A)] = 0. \tag{68}$$

The determinant of this set of equations should be 0, for a solution of the B.H.-equations to exist in the neighbourhood of N_{cr}, which condition results in:

$$N[1 - \cos(A)] - 2 = 0. \tag{69}$$

Together with (58) this will give us the value of N_{cr} and the corresponding A.

In an analogous way we find for the case II the following set of equations as a result of linearizing (62),(64) and (66) in terms of b, p and r:

$$Nb - p - r = 0, \tag{70}$$

$$b - \frac{p}{2}[1 - \cos(A)] = 0, \tag{71}$$

$$b[1 - \cos(n)][1 + \cos(R)] - r[1 - \cos(A)] = 0. \tag{72}$$

Putting the value of the determinant of this set equal to 0 and combining this condition with (61), (63) and (65) we find N_{cr} and the corresponding values of A and n. In equation (65) we have to substitute the value 0 for b and r.

In this way we found the following critical values:

$$\text{case I: } N_{cr} = 61.3488 \qquad \text{case II: } N_{cr} = 62.3514. \tag{73}$$

Below the critical point the quantities b, p and r take purely imaginary values, and e.g. (60) should be replaced by:

$$\cosh(\frac{p}{2})\sinh(b) - \sinh(\frac{p}{2})[\cosh(b) - \cos(A)] = 0. \tag{74}$$

c) Limit point

The simplest example of such a point may be found in subsection 6.1. We now take a fixed value for the $\{\lambda_j\}$, e.g. $\lambda_1 = \lambda_2 = 4$ and follow the solution in lowering N. Now it turns out that b as well as q have the limit ∞ for N approaching the value 16, as a consequence of the fact that:

$$\cos(\frac{\pi\lambda}{N}) = 0 \text{ for } \lambda = 2\lambda_1 = 8 \text{ and } \ N_{\text{lim}} = 16. \tag{75}$$

For lower values of N the change of sign of $\cos(\pi\lambda/N)$ results in the vanishing of this solution. Analogous limit points may appear the cases in which there is a pair $\lambda_j = \lambda_{j+1} - 1$. We here consider the case I analyzed in Appendix b, which results in the equation (74), with $A = -5\pi/N$ and $b = p/N$, which gives $b = p = \infty$ for the limit value $N_{\text{lim}} = 10$.

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